Homework III

Due: Dec. 7. (Tue) 23:59 PM

I. REMARK

- Reading materials: Ch 1-6 in the textbook.
- Don't write just an answer. Please describe enough processes to justify your answer (Korean is also OK!!).
- Better the last smile than the first laughter.

II. PROBLEM SET

1) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T: M_{2\times 2} \to M_{2\times 2}$ by $T(A) = A + A^T$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   a) Show that T is a linear transformation.
   b) Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^T = B$.
   c) Describe the kernel of T.

2) Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x_1 + 2x_2 + x_3 = 0$.

3) Let $\mathbb{P}_2$ be the set of polynomials where the degree of polynomials is 2.
   a) The set $B = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the change-of-coordinates matrix from the basis B to the standard basis $C = \{1, t, t^2\}$.
   b) Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to B.

4) Show that the set of polynomials $\{1, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$ is a basis of $\mathbb{P}_3$.

5) Diagonalize the matrix if possible.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

6) Assume the mapping $T: \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$ is linear. Find the matrix representation of T relative to the basis $B = \{1, t, t^2\}$.

7) Let $T: \mathbb{P}_2 \to \mathbb{P}_4$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $\mathbf{p}(t) + t^2\mathbf{p}(t)$.
   a) Find the image of $\mathbf{p}(t) = 2 - t + t^2$.
   b) Show that T is a linear transformation.
   c) Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

8) Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$.
   a) Show that A(Re $\mathbf{v}$) = a(Re $\mathbf{v}$) + b(Im $\mathbf{v}$) and A(Im $\mathbf{v}$) = $-b$(Re $\mathbf{v}$) + a(Im $\mathbf{v}$).
   b) Verify that if P and C are given as in Theorem 9 in Ch. 5.5, then $AP = PC$.

9) Classify the origin as an attractor, repeller, or saddle point of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. A is given as

$$A = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix}$$

10) Let $\mathbf{y}= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthonal to $\mathbf{u}$.

11) Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Also, find the distance from $\mathbf{z}$ to the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

12) Find an orthogonal basis for the column space of the matrix (Use the Gram-Schmidt process).

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

13) Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

14) Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points. $\{(0,1), (1,1), (2,2), (3,2)\}$.

15) For x and y in $\mathbb{P}_3$, define $< x, y >= x(-3)y(-3) + x(-1)y(-1) + x(1)y(1) + x(3)y(3)$. Let $p_0(t) = 1$, $p_1(t) = t$, and $p_2(t) = t^2$.

a) Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.

b) Find a polynomial q that is orthogonal to p_0 and p_1, such that $\{p_0, p_1, q\}$ is an orthogonal basis for Span $\{p_0, p_1, p_2\}$. Scale the polynomial q so that its vector of values at $(-3, -1, 1, 3)$ is $(1, -1, -1, 1)$.