Midterm Exam

Engineering Mathematics, Fall 2021
School of BioMedical Convergence Engineering, PNU
Oct. 19. 10:00 - 12:00

I. REMARK

- This is a closed book exam. You are permitted on two pages of notes.
- There are a total of 100 points in the exam. Each problem specifies its point total.
- You must SHOW YOUR WORK to get full credit.

II. PROBLEM SET

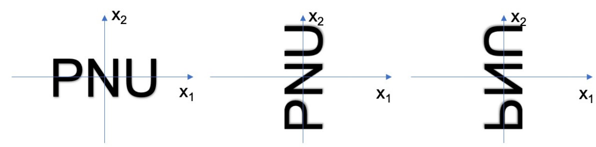
1) [20 points] Mark each statement True or False. You don't need to justify each answer in this problem.
   a) If the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions, $\mathbf{b}$ is in the set spanned by the columns of A. [True/False]
   b) If $A \in \mathbb{R}^{n\times n}$ and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] \in \mathbb{R}^{n\times n}$, then $BA = [A\mathbf{b}_1\ A\mathbf{b}_2\ A\mathbf{b}_3]$. [True/False]
   c) If A is invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$. [True/False]
   d) If A is a 3×4 matrix, then the transformation $\mathbf{x} \to A\mathbf{x}$ cannot be one-to-one.[True/False]
   e) The set of columns of $A \in \mathbb{R}^{m\times n}$ is independent, then every row of A has a pivot. [True/False]
   f) The dimension of Col A is the number of pivot columns of A. [True/False]
   g) A is an $n \times n$ matrix. The eigenvalues of A are on its main diagonal. [True/False]
   h) If the set of columns of $A \in \mathbb{R}^{n\times n}$ is linearly dependent, then det$A = 0$. [True/False]
   i) If H is a p-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of p vectors in H is a basis for H.
   j) The dimension of Nul A is the number of nonpivot columns in A. [True/False]

2) [20 points] Let $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -2 & 3 & 0 & 1 \\ -1 & 2 & 4 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.$$

   a) Find a solution set $\mathbf{x}$ if it is consistent.
   b) Find a basis of Col$A$.
   c) Find a basis of Nul$A$.
   d) Find an LU factorization of the matrix A. Note that $L \in \mathbb{R}^{3\times 3}$ and $U \in \mathbb{R}^{3\times 4}$.

3) [15 points] $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $\pi/2$ radian, counter-clockwise ((A) to (B)) and then reflects points through the vertical x_2-axis ((B) to (C)).
   a) Find the standard matrix of T ((A) to (C)).
   b) Derive that T^m . $\begin{cases} I & \text{if } m \text{ is even} \\ T & \text{otherwise} \end{cases}$



(A) (B) (C)

4) [10 points] A set $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and a vector $\mathbf{a}$ are given as

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix},$$

   a) Is $\mathbf{a}$ in span B?
   b) Find the solution of $B\mathbf{x} = \mathbf{a}$ using Cramer's rule.

5) [10 points] Suppose that A is a 3×3 matrix with the property that $A\mathbf{v}_1 = \mathbf{e}_1$, $A\mathbf{v}_2 = \mathbf{e}_2$ and $A\mathbf{v}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 13 \\ 15 \\ 17 \end{bmatrix}.$$

Find A^{-1}. ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ denote the standard basis vectors in $\mathbb{R}^3$)

6) [15 points] A matrix A is given as

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

   a) Show that $A + A^{-1} = 2I$.
   b) Find A^{100} using a).
   c) Find one eigenvalue of A. Find the eigenspace corresponding to the eigenvalue.

7) [10 points] if the columns of B are linearly dependent, are the columns of AB are linearly dependent? Why?