Final Exam

BX25203-141 Engineering Mathematics (I), Fall 2021
School of BioMedical Convergence Engineering, PNU
Dec. 14. 10:30 - 13:00

I. REMARK

- This is a closed book exam. You are permitted on three pages of notes.
- There are a total of 100 points in the exam. Each problem specifies its point total.
- You must SHOW YOUR WORK to get full credit.

II. PROBLEM SET

1) [10 points] Mark each statement True or False. You don't need to justify each answer in this problem (Let $\mathbb{P}_n$ be the set of polynomials where the degree is n.)
   a) $\mathbb{R}^2$ is the subspace of $\mathbb{R}^4$. [True/False]
   b) Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$, $\mathbf{p}_2(t) = 1 - t^2$ and $\mathbf{p}_3(t) = t^3$. Then, $\{\mathbf{p}_1, \mathbf{p}_2\, \mathbf{p}_3\}$ is a basis for $\mathbb{P}_3$. [True/False]
   c) If A is similar to B, then A^2 is similar to B^2. [True/False]
   d) If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$. [True/False]
   e) Let λ be an eigenvalue of an invertible matrix A. Then λ^{-1} is an eigenvalue of A^{-1}. [True/False]
   f) If $\mathbf{x}$ is in a subspace W, then $\mathbf{x} - \mathrm{proj}_W \mathbf{x}$ is not zero vector. [True/False]
   g) The rows of $A \in \mathbb{R}^{n \times n}$ span $\mathbb{R}^n$ if and only if A has n pivot positions. [True/False]
   h) If $A \in \mathbb{R}^{n \times n}$ is diagonalizable, then A has n distinct eigenvalues. [True/False]
   i) For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.[True/False]

2) [5 points] Let $D = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$
   a) Find the change-of-coordinates matrix from F to D.
   b) Find $[\mathbf{x}]_D$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

3) [5 points] Find the least-squares line $y = \alpha_0 + \alpha_1 x$ that best fits the data $(-2, 3)$, $(-1, 5)$,$(0, 5)$, $(1, 4)$ and $(2, 3)$.

4) [10 points] The mapping T: $\mathbb{P}_3 \to \mathbb{P}_2$ is derivation defined by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_2 t^2$$

   B is the basis $\{2, 1 + t, t + t^2, t^3\}$ for $\mathbb{P}_3$ and C is the basis $\{1, 2t, 2t^2\}$ for $\mathbb{P}_2$.
   a) Find the matrix for T relative to the bases B and C.
   b) Find the image under T of $p(t) = 1 + 2t + 3t^2$.

5) [10 points] The matrix A is given as

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

   a) Diagonalize the matrix so that $A = PDP^{-1}$.
   b) Basis B is formed from the columns of P. If $[\mathbf{x}]_B = [1, 1, 1]^T$, what is $A\mathbf{x}$?

6) [15 points] The linear equation is given as $A\mathbf{x} = \mathbf{y}$ where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

   a) Solve the linear equation. Is the equation consistent?
   b) Find an orthogonal basis for the column space of A.
   c) Find the $\hat{\mathbf{y}} = \mathrm{Proj}_{\mathrm{Col}A}\mathbf{y}$ using b).
   d) Find the solution of $\hat{\mathbf{y}} = A\mathbf{x}$. Explain why the equation must be consistent.
   e) Find the least-square solution using the normal equation. Check that it is same to the solution of d).

7) [10 points] For x and y in $\mathbb{P}_3$, define $< x, y >= x(-3)y(-3) + x(-1)y(-1) + x(1)y(1) + x(3)y(3)$. Let $q(t) = t^3 + t^2$.
   a) Compute the orthogonal projection of $q(t)$ onto the subspace $\mathbb{P}_2$.
   b) Find the $g(t) \in \mathbb{P}_1$ such that $||g(t) - q(t)||$ is minimized.

8) [10 points] One institution inspects the average financial state of PNU students. Let x_k be the average income at year k, and y_k be the average debt. Assume that

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.6 \\ -0.3 & 1.4 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 100,000,000 \\ 100,000,000 \end{bmatrix},$$

Explain what happens when $k \to \infty$. What is the ratio x_k/y_k when $k \to \infty$?

9) [10 points] If A is 3×3 symmetric positive definite, then $A\mathbf{q}_i = \lambda_i \mathbf{q}_i$ with positive eigenvalues λ_i and orthonormal eigenvectors $\mathbf{q}_i$. Let $\lambda_1 > \lambda_2 > \lambda_3$. Suppose $\mathbf{x} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3$.
   a) Compute $\mathbf{x}^T\mathbf{x}$ and $\mathbf{x}^T A\mathbf{x}$ in terms of the c's and λs.
   b) Find $\mathbf{x}$ which maximize the ratio $\mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$. Explain the reason of your answer in detail.

10) [15 points] Suppose $A = PDP^{-1} \in \mathbb{R}^n$ is symmetry. The eigenvectors of A are $\mathbf{u}_1, \mathbf{u}_1, \ldots, \mathbf{u}_n$, and the corresponding eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$.
   a) Show that $A = \lambda_1\mathbf{u}_1\mathbf{u}_1^T + \lambda_2\mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^T$.
   b) Suppose $A = A^2$. Then, what is the condition in terms of $\lambda_1 \cdots \lambda_n$?
   c) Suppose $A = A^2$. Given any $\mathbf{y} \in \mathbb{R}^n$, let $\hat{\mathbf{y}} = A\mathbf{y}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Show that $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.